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NONLINEAR SYSTEMS

Towards Computation of Surface Area of Schur Stability Domain

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Abstract—The paper considers the subset of the Schur stability domain, namely, the set of parameters for which the roots of polynomials of degree n lie in the complex unit disc and are real numbers. The method for computation of the hypersurface area of this defined set in n-dimensional space is obtained. The maximum surface area of this set is reached for $n = 3$, i.e., 3-dimensional set has maximum surface area.

Keywords: multiple integrals, Schur stability domain, autoregression, stationary autoregressive processes, unit root processes

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1. INTRODUCTION

A significant part of recent research was devoted to the study of Schur stability domain and its boundary [1]. Though the question of determining the size of this domain's boundary, i.e. domain's hypersurface area, is still open.

We study polinomials of the form

$$x^n - \alpha_1 x^{n-1} - \ldots - \alpha_{n-1} x - \alpha_n = 0, \tag{1}$$

with parameters $\alpha_i \in R,\ i = 1, \ldots, n$.

Each polinomial corresponds to a single point $(\alpha_1, \ldots, \alpha_n)$ in n-dimensional Euclidean space R^n.

Schur stability domain is defined as a set in R^n, for which roots of corresponding polinomials lie in unit disc on the complex plane. We denote this set as D_n, as in [2]. That paper provides formula for computation of the volume $V(D_n)$.

We denote as $\mathcal{E}_n$, as in [3], the subset of D_n where roots of polinomials are real numbers with zero imaginary part. That paper provides formula for computation of the volume $V(\mathcal{E}_n)$ with a multiple integral, that is related to the Selberg type integral.

Volumes of stability domains [2, 3]

n	$V(D_n)$	$V(\mathcal{E}_n)$	$V(D_n \backslash \mathcal{E}_n)$	$\frac{V(\mathcal{E}_n)}{V(D_n)}$	$\frac{V(D_n \backslash \mathcal{E}_n)}{V(D_n)}$
“1”	“2”	“3”	“4”	“5”	“6”
1	2	2	0	1	0
2	4	1.333	2.667	0.3333	0.6667
3	5.333	0.355	4.978	0.0667	0.9333
4	7.111	0.041	7.070	0.0057	0.9943
5	7.585	0.002	7.583	0.0003	0.9997

Volumes $V(D_n)$ and $V(\mathcal{E}_n)$ reach maximum with $n = 6$ and $n = 1$ respectively and are infinitesimal as $n \to \infty$.[1]

The boundary ∂D_n of domain D_n is the union of two hyperplanes and one hypersurface [1, 4]. The hyperplanes correspond to roots -1 and 1.

The boundary $\partial \mathcal{E}_n$ of domain $\mathcal{E}_n$ consists of two hyperplanes, that correspond to hyperplanes of ∂D_n, and one hypersurface inside of D_n.

This paper presents results of computation of hypersurface area $S(\partial \mathcal{E}_n)$ of the boundary $\partial \mathcal{E}_n$ of the domain $\mathcal{E}_n$.

2. COMPUTATION OF $S(\partial \mathcal{E}_2)$

Consider polinom of degree 2:

$$x^2 - \alpha_1 x - \alpha_2 = 0. \tag{2}$$

Domain D_2 is the set of parameters:

$$\begin{cases} -2 < \alpha_1 < 2 \\ -1 < \alpha_2 < 1 - |\alpha_1|. \end{cases}$$

Domain $\mathcal{E}_2$ is the set of parameters:

$$\begin{cases} -2 < \alpha_1 < 2 \\ -\alpha_1^2/4 < \alpha_2 < 1 - |\alpha_1|. \end{cases}$$

From econometric point of view the picture shows stationarity domains of parameters that correspond to stationary autoregressive processes of the second order, while the outer boundary corresponds to unit root processes and the outer region corresponds to explosive processes.

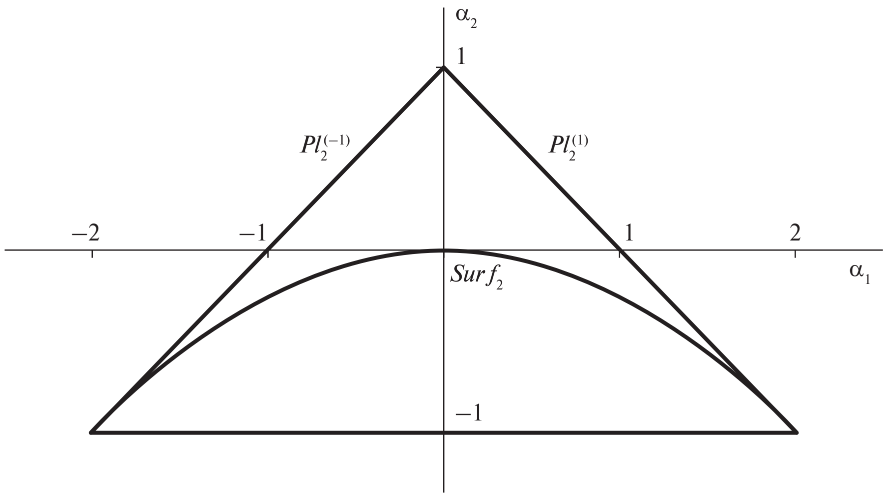

Fig. 1. Domains D_2 and $\mathcal{E}_2$. Source: [5].

[1] Though comparing values of not equal dimensions, such as length, area, volume, etc. may seem strange enough, sometimes it may also be curious. For instance, see study of these issues for hyperball and hypersphere:
https://mathworld.wolfram.com/Hypersphere.html
https://mathworld.wolfram.com/Ball.html

The boundary $\partial\mathcal{E}_2$ consists of parabola section $Surf_2$ and two line segments $Pl_2^{(1)}$ and $Pl_2^{(-1)}$, that correspond to roots 1 and −1. I.e. polinom (2) with parameters from $Pl_2^{(1)}$ has root, equal to 1, and polinom (2) with parameters from $Pl_2^{(-1)}$ has root, equal to −1.

The boundary length $S(\partial\mathcal{E}_2)$ can be computed as a sum of lengths:

$$S(\partial\mathcal{E}_2) = S(Pl_2^{(1)}) + S(Pl_2^{(-1)}) + S(Surf_2) \approx 2\sqrt{2} + 2\sqrt{2} + 4.591 \approx 10.248.$$

3. COMPUTATION OF $S(\partial\mathcal{E}_3)$

Consider polinom of degree 3:

$$x^3 - \alpha_1 x^2 - \alpha_2 x - \alpha_3 = 0. \tag{3}$$

According to [6], domain D_3 is a set of parameters:

$$\begin{cases} -1 < \alpha_3 < 1 \\ -3 < \alpha_1 < 3 \\ \alpha_2 < 1 - |\alpha_1 + \alpha_3| \\ \alpha_2 > -1 + \alpha_3^2 - \alpha_1\alpha_3. \end{cases}$$

Domain $\mathcal{E}_3$ is a set of parameters:

$$\begin{cases} -1 < \alpha_3 < 1 \\ -3 < \alpha_1 < 3 \\ \alpha_2 < 1 - |\alpha_1 + \alpha_3| \\ \alpha_1^2\alpha_2^2 + 4\alpha_2^3 - 4\alpha_1^3\alpha_3 - 27\alpha_3^2 - 18\alpha_1\alpha_2\alpha_3 > 0. \end{cases}$$

The boundary $\partial\mathcal{E}_3$ consists of the surface $Surf_3$ and two plane sections $Pl_3^{(1)}$ and $Pl_3^{(-1)}$, that correspond to roots 1 and −1. I.e. polinom (3) with parameters from section $Pl_3^{(1)}$ has a root, equal to 1, and polinom (3) with parameters from section $Pl_3^{(-1)}$ has a root, equal to −1. These surfaces can be parameterized applying Vieta's formulas.

Parametric form of $Pl_3^{(1)}$:

$$\begin{cases} \alpha_1 = 1 + x_2 + x_3 \\ \alpha_2 = -x_2 - x_3 - x_2x_3 \\ \alpha_3 = x_2x_3, \end{cases}$$

where roots of polinomial (3) $x_1 = 1$, $x_2 \in [-1, 1]$, $x_3 \in [-1, x_2]$.

Parametric form of $Pl_3^{(-1)}$:

$$\begin{cases} \alpha_1 = -1 + x_2 + x_3 \\ \alpha_2 = x_2 + x_3 - x_2x_3 \\ \alpha_3 = -x_2x_3, \end{cases}$$

where roots of polinomial (3) $x_1 = -1$, $x_2 \in [-1, 1]$, $x_3 \in [-1, x_2]$.

Parametric form of $Surf_3$:

$$\begin{cases} \alpha_1 = 2t + x_3 \\ \alpha_2 = -t^2 - 2tx_3 \\ \alpha_3 = t^2x_3, \end{cases} \tag{4}$$

where roots of polinomial (3) $x_1 = x_2 = t$, $t \in [-1, 1]$, $x_3 \in [-1, 1]$.

The boundary area $S(\partial\mathcal{E}_3)$ can be computed as a sum of surface areas:

$$S(\partial\mathcal{E}_3) = S(Pl_3^{(1)}) + S(Pl_3^{(-1)}) + S(Surf_3) \approx \sqrt{3} \times \frac{4}{3} + \sqrt{3} \times \frac{4}{3} + 6.759 \approx 11.378. \quad (5)$$

4. COMPUTATION OF $S(\partial\mathcal{E}_n)$

Consider polinom of degree n in the form (1).

The boundary $\partial\mathcal{E}_n$ of domain $\mathcal{E}_n$ consists of hypersurface $Surf_n$ and two hyperplanes sections $Pl_n^{(1)}$ and $Pl_n^{(-1)}$, that correspond to roots 1 and -1, i.e., polinom (1) with parameters from section $Pl_n^{(1)}$ has a root, equal to 1, and polinom (1) with parameters from section $Pl_n^{(-1)}$ has a root, equal to -1.

Statements about hypersurface areas of hyperplanes sections $Pl_n^{(1)}$ and $Pl_n^{(-1)}$ as well as of hypersurface $Surf_n$ we formulate as propositions.

Proposition 1. *Hypersurface area of hyperplanes sections $Pl_n^{(1)}$ and $Pl_n^{(-1)}$, $n \geqslant 2$, can be computed as*

$$S(Pl_n^{(1)}) = S(Pl_n^{(-1)}) = \sqrt{n} \times 2^{\frac{n(n-1)}{2}} \prod_{k=1}^{n-1} \frac{\{(k-1)!\}^2}{(2k-1)!}. \quad (6)$$

Proposition 2. *Hypersurface area of hypersurface $Surf_n$, $n \geqslant 3$, can be computed as*

$$S(Surf_n) = 2 \int\limits_{\substack{x_1 \in (-1,1) \\ -1 \leqslant x_2 \leqslant \ldots \leqslant x_{n-1} \leqslant 1}} \cdots \int \frac{\sqrt{1 - x_1^{2n}}}{\sqrt{1 - x_1^2}} \prod_{1 \leqslant i < j \leqslant n-1} |x_i - x_j| \, dx_1 \ldots dx_{n-1}. \quad (7)$$

In particular, for $n = 3$

$$S(Surf_3) = 2 \iint\limits_{\substack{-1 < x_1 < 1 \\ -1 \leqslant x_2 \leqslant 1}} \frac{\sqrt{1 - x_1^6}}{\sqrt{1 - x_1^2}} |x_1 - x_2| \, dx_1 dx_2 \approx 6.759.$$

The boundary area $S(\partial\mathcal{E}_n)$ can be computed as a sum of hypersurface areas:

$$S(\partial\mathcal{E}_n) = S(Pl_n^{(1)}) + S(Pl_n^{(-1)}) + S(Surf_n). \quad (8)$$

Next proposition presents the main result of the paper.

Proposition 3. *Hypersurface area $S(\partial\mathcal{E}_n)$ is infinitesimal with $n \to \infty$. The maximum value is reached for $n = 3$.*

5. CONCLUSION

The paper presents the results of computation of hypersurface area $S(\partial\mathcal{E}_n)$ of the boundary of the domain $\mathcal{E}_n$ in n-dimensional space, that is a subset of Schur stability domain D_n.

As a topic for further research we suggest the computation of hypersurface area $S(\partial D_n)$ of the boundary of the domain D_n. As already mentioned, boundary ∂D_n consists of two hyperplanes sections $Pl_n^{(1)}$ and $Pl_n^{(-1)}$, that correspond to roots 1 and -1, and one hypersurface, that is denoted as $\mathfrak{S}_n$. Computation of hypersurface area $S(\mathfrak{S}_n)$ for arbitrary n seems to be a more difficult task, comparing with the result (7) in Proposition 2.

APPENDIX

A.1. PROOF OF PROPOSITION 1

With standard methods one can check, that (6) holds for $n = 2, 3$. On the basis of (6) one can obtain

$$S(Pl_2^{(1)}) = 2\sqrt{2},$$
$$S(Pl_3^{(1)}) = \sqrt{3}\,\frac{4}{3},$$

that agrees with the values, computed in Sections 2 and 3.

Now we will prove (6) for arbitrary n.

The hyperplane section $Pl_n^{(1)}$ can be parameterized:

$$\begin{cases} \alpha_1 = 1 + \zeta_1 \\ \alpha_2 = -(\zeta_1 + \zeta_2) \\ \alpha_3 = \zeta_2 + \zeta_3 \\ \dots \\ \alpha_k = (-1)^{k-1}(\zeta_{k-1} + \zeta_k) \\ \dots \\ \alpha_{n-2} = (-1)^{n-3}(\zeta_{n-3} + \zeta_{n-2}) \\ \alpha_{n-1} = (-1)^{n-2}(\zeta_{n-2} + \zeta_{n-1}) \\ \alpha_n = (-1)^{n-1}\zeta_{n-1}, \end{cases} \tag{A.1}$$

where $\zeta_i,\ i = 1, \dots, n-1$ – elementary symmetric polinomial of degree i of $(n-1)$ variables $x_2, \dots, x_n$, $-1 \leqslant x_2 \leqslant \dots \leqslant x_n \leqslant 1$.

Parametric form (A.1) is derived using Vieta's formulas and taking $x_1 = 1$. Ordering of parameters $x_2, \dots, x_n$ is caused by the fact, that parametric form (A.1) is symmetric with respect to the parameters permutations.

By definition

$$S(Pl_n^{(1)}) = \underset{-1\leqslant x_2 \leqslant \dots \leqslant x_n \leqslant 1}{\int \cdots \int} \|H_n\| dx_2 \dots dx_n, \tag{A.2}$$

where H_n – determinant:

$$H_n = \begin{vmatrix} \mathrm{j}_1 & \dfrac{\partial \alpha_1}{\partial x_2} & \dfrac{\partial \alpha_1}{\partial x_3} & \cdots & \dfrac{\partial \alpha_1}{\partial x_n} \\ \mathrm{j}_2 & \dfrac{\partial \alpha_2}{\partial x_2} & \dfrac{\partial \alpha_2}{\partial x_3} & \cdots & \dfrac{\partial \alpha_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{j}_n & \dfrac{\partial \alpha_n}{\partial x_2} & \dfrac{\partial \alpha_n}{\partial x_3} & \cdots & \dfrac{\partial \alpha_n}{\partial x_n} \end{vmatrix}, \tag{A.3}$$

where $\{\mathrm{j}_1, \dots, \mathrm{j}_n\}$ – orthonormal basis in R^n.

For ζ_i from (A.1) we have

$$\frac{\partial \zeta_i}{\partial x_r} = \zeta_{i-1}^{(r)},$$

where $\zeta_j^{(r)}$– elementary symmetric polinomial of degree j of $n-2$ variables $x_2, \dots, x_{r-1}, x_{r+1}, \dots, x_n$.

Without loss of generality let $n = 2k + 1$. Then on the basis of (A.3) and (A.1) one can obtain

$$H_n = \begin{vmatrix} \mathrm{j}_1 & 1 & 1 & \cdots & 1 \\ \mathrm{j}_2 & -1-\zeta_1^{(2)} & -1-\zeta_1^{(3)} & \cdots & -1-\zeta_1^{(n)} \\ \mathrm{j}_3 & \zeta_1^{(2)}+\zeta_2^{(2)} & \zeta_1^{(3)}+\zeta_2^{(3)} & \cdots & \zeta_1^{(n)}+\zeta_2^{(n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{j}_{n-1} & -\zeta_{n-3}^{(2)}-\zeta_{n-2}^{(2)} & -\zeta_{n-3}^{(3)}-\zeta_{n-2}^{(3)} & \cdots & -\zeta_{n-3}^{(n)}-\zeta_{n-2}^{(n)} \\ \mathrm{j}_n & \zeta_{n-2}^{(2)} & \zeta_{n-2}^{(3)} & \cdots & \zeta_{n-2}^{(n)} \end{vmatrix}. \tag{A.4}$$

Consider $H_n^{(n)}$ – cofactor of j_n in expansion of H_n along the first column:

$$H_n^{(n)} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ -1-\zeta_1^{(2)} & -1-\zeta_1^{(3)} & \cdots & -1-\zeta_1^{(n)} \\ \zeta_1^{(2)}+\zeta_2^{(2)} & \zeta_1^{(3)}+\zeta_2^{(3)} & \cdots & \zeta_1^{(n)}+\zeta_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ -\zeta_{n-3}^{(2)}-\zeta_{n-2}^{(2)} & -\zeta_{n-3}^{(3)}-\zeta_{n-2}^{(3)} & \cdots & -\zeta_{n-3}^{(n)}-\zeta_{n-2}^{(n)} \end{vmatrix}.$$

With standard transformations of rows $H_n^{(n)}$ can be transformed to the form

$$H_n^{(n)} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ -\zeta_1^{(2)} & -\zeta_1^{(3)} & \cdots & -\zeta_1^{(n)} \\ \zeta_2^{(2)} & \zeta_2^{(3)} & \cdots & \zeta_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ -\zeta_{n-2}^{(2)} & -\zeta_{n-2}^{(3)} & \cdots & -\zeta_{n-2}^{(n)} \end{vmatrix}$$

While considering hyperplane section $Pl_n^{(1)}$ with $\sum\limits_{i=1}^{n} \alpha_i = 1$ in (A.1), we have $\sum\limits_{i=1}^{n} \frac{\partial \alpha_i}{\partial x_j} = 0$ for all $j = 2, \ldots, n$.

Thus, all cofactors in expansion of H_n along the first column are equal in absolute values, hence it is sufficient to evaluate one of them, for instance $H_n^{(n)}$.

Then on the basis of (A.2) and [3] one can derive

$$S(Pl_n^{(1)}) = \sqrt{n} \underset{-1\leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant 1}{\int \cdots \int} |H_n^{(n)}| dx_2 \ldots dx_n = \sqrt{n}\, V(\mathcal{E}_{n-1}), \tag{A.5}$$

where $V(\mathcal{E}_{n-1})$ – volume of domain $\mathcal{E}_{n-1}$.

To complete the proof for $Pl_n^{(1)}$ it remains to take into account the result from [3], that

$$V(\mathcal{E}_n) = 2^{\frac{n(n+1)}{2}} \prod_{k=1}^{n} \frac{\{(k-1)!\}^2}{(2k-1)!}.$$

The proof for $Pl_n^{(-1)}$ is obtained in the same way.

The proof of Proposition 1 is finished.

A.2. PROOF OF PROPOSITION 2

We will prove the proposition by induction.

Let $n = 3$. Surface $Surf_3$ is parameterized in (4). One can calculate its area $S(Surf_3)$ in the following way:

$$S(Surf_3) = \iint\limits_{\substack{-1\leqslant t\leqslant 1\\ -1\leqslant x_3\leqslant 1}} \sqrt{\left(\frac{D(\alpha_1,\alpha_2)}{D(t,x_3)}\right)^2 + \left(\frac{D(\alpha_1,\alpha_3)}{D(t,x_3)}\right)^2 + \left(\frac{D(\alpha_2,\alpha_3)}{D(t,x_3)}\right)^2}\, dt\, dx_3,$$

where

$$\frac{D(\alpha_1,\alpha_2)}{D(t,x_3)} = \begin{vmatrix} \dfrac{\partial\alpha_1}{\partial t} & \dfrac{\partial\alpha_1}{\partial x_3} \\ \dfrac{\partial\alpha_2}{\partial t} & \dfrac{\partial\alpha_2}{\partial x_3} \end{vmatrix} \simeq 2(t - x_3), \qquad \frac{D(\alpha_1,\alpha_3)}{D(t,x_3)} = \begin{vmatrix} \dfrac{\partial\alpha_1}{\partial t} & \dfrac{\partial\alpha_1}{\partial x_3} \\ \dfrac{\partial\alpha_3}{\partial t} & \dfrac{\partial\alpha_3}{\partial x_3} \end{vmatrix} \simeq 2t(t - x_3),$$

$$\frac{D(\alpha_2,\alpha_3)}{D(t,x_3)} = \begin{vmatrix} \dfrac{\partial\alpha_2}{\partial t} & \dfrac{\partial\alpha_2}{\partial x_3} \\ \dfrac{\partial\alpha_3}{\partial t} & \dfrac{\partial\alpha_3}{\partial x_3} \end{vmatrix} \simeq 2t^2(t - x_3).$$

The sign "$\simeq$" means being equal in absolute values. Its usage is convenient and valid, because corresponding values will be squared at the right moment.

Thus,

$$S(Surf_3) = \iint\limits_{\substack{-1\leqslant t\leqslant 1\\ -1\leqslant x_3\leqslant 1}} \|A_3\|\, dt\, dx_3, \tag{A.6}$$

where

$$\|A_3\| = 2\sqrt{1 + t^2 + t^4}\ |t - x_3|. \tag{A.7}$$

The induction base is (A.6) for $S(Surf_3)$.

Now consider hypersurface $Surf_n$ for arbitrary n, that is parameterized as:

$$\begin{cases} \alpha_1 = 2t + \sigma_1 \\ \alpha_2 = -(t^2 + 2t\sigma_1 + \sigma_2) \\ \alpha_3 = t^2\sigma_1 + 2t\sigma_2 + \sigma_3 \\ \dots \\ \alpha_k = (-1)^{k-1}(t^2\sigma_{k-2} + 2t\sigma_{k-1} + \sigma_k) \\ \dots \\ \alpha_{n-2} = (-1)^{n-3}(t^2\sigma_{n-4} + 2t\sigma_{n-3} + \sigma_{n-2}) \\ \alpha_{n-1} = (-1)^{n-2}(t^2\sigma_{n-3} + 2t\sigma_{n-2}) \\ \alpha_n = (-1)^{n-1}t^2\sigma_{n-2}, \end{cases} \tag{A.8}$$

where $\sigma_i,\ i = 1, \dots, n-2,$ – elementary symmetric polinomial of degree i of $(n-2)$ variables $x_3, \dots, x_n,\ \ t \in [-1, 1],\ \ -1 \leqslant x_3 \leqslant \dots \leqslant x_n \leqslant 1$.

Parametric form (A.8) is derived on the basis of Vieta's formulas after equating $x_1 = x_2 = t$. Ordering of parameters $x_3, \dots, x_n$ is caused by the fact, that parametric form (A.8) is symmetric with respect to the parameters permutations.

The induction assumption is that the proposition is true for hypersurface area $S(Surf_n)$. I.e. we assume, that

$$S(Surf_n) = \int\limits_{\substack{t\in[-1,1]\\ -1\leqslant x_3\leqslant\ldots\leqslant x_n\leqslant 1}} \cdots \int \|A_n\| dt\, dx_3 \ldots dx_n, \tag{A.9}$$

where

$$\|A_n\| = 2\sqrt{1+t^2+\ldots+t^{2(n-1)}}\left(\prod_{3\leqslant i\leqslant n} |t-x_i|\right)\left(\prod_{3\leqslant i<j\leqslant n} |x_i-x_j|\right). \tag{A.10}$$

One can note, that formulas (A.9) and (7) are equivalent, but for current purposes (A.9) is more convenient.

Now on the basis of (A.10) we will prove the proposition for hypersurface area $S(Surf_{n+1})$.

Let $y = x_{n+1}$. Then hypersurface $Surf_{n+1}$ is parameterized in the following way:

$$\begin{cases} \alpha_1 = 2t + \sigma_1 + y \\ \alpha_2 = -(t^2 + 2t(\sigma_1 + y) + \sigma_2 + y\sigma_1) \\ \alpha_3 = t^2(\sigma_1 + y) + 2t(\sigma_2 + y\sigma_1) + \sigma_3 + y\sigma_2 \\ \alpha_4 = -(t^2(\sigma_2 + y\sigma_1) + 2t(\sigma_3 + y\sigma_2) + \sigma_4 + y\sigma_3) \\ \ldots \\ \alpha_k = (-1)^{k-1}(t^2(\sigma_{k-2} + y\sigma_{k-3}) + 2t(\sigma_{k-1} + y\sigma_{k-2}) + \sigma_k + y\sigma_{k-1}) \\ \ldots \\ \alpha_{n-1} = (-1)^{n-2}(t^2(\sigma_{n-3} + y\sigma_{n-4}) + 2t(\sigma_{n-2} + y\sigma_{n-3}) + y\sigma_{n-2}) \\ \alpha_n = (-1)^{n-1}(t^2(\sigma_{n-2} + y\sigma_{n-3}) + 2ty\sigma_{n-2}) \\ \alpha_{n+1} = (-1)^n t^2 y\sigma_{n-2}, \end{cases} \tag{A.11}$$

where σ_i, $i = 1,\ldots,n-2$, were defined in (A.8), $t \in [-1,1]$, $-1 \leqslant x_3 \leqslant \ldots \leqslant x_n \leqslant y \leqslant 1$.

Parametric form (A.11) is derived on the basis of Vieta's formulas after equating $x_1 = x_2 = t$. Ordering of parameters $x_3,\ldots,x_n,y$ is caused by the fact, that parametric form (A.11) is symmetric with respect to the parameters permutations.

By definition

$$S(Surf_{n+1}) = \int\limits_{\substack{t\in[-1,1]\\ -1\leqslant x_3\leqslant\ldots\leqslant x_n\leqslant y\leqslant 1}} \cdots \int \|A_{n+1}\| dt\, dx_3 \ldots dx_n\, dy, \tag{A.12}$$

where A_{n+1} – determinant:

$$A_{n+1} = \begin{vmatrix} \mathrm{j}_1 & \frac{\partial\alpha_1}{\partial t} & \frac{\partial\alpha_1}{\partial x_3} & \cdots & \frac{\partial\alpha_1}{\partial x_n} & \frac{\partial\alpha_1}{\partial y} \\ \mathrm{j}_2 & \frac{\partial\alpha_2}{\partial t} & \frac{\partial\alpha_2}{\partial x_3} & \cdots & \frac{\partial\alpha_2}{\partial x_n} & \frac{\partial\alpha_2}{\partial y} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathrm{j}_n & \frac{\partial\alpha_n}{\partial t} & \frac{\partial\alpha_n}{\partial x_3} & \cdots & \frac{\partial\alpha_n}{\partial x_n} & \frac{\partial\alpha_n}{\partial y} \\ \mathrm{j}_{n+1} & \frac{\partial\alpha_{n+1}}{\partial t} & \frac{\partial\alpha_{n+1}}{\partial x_3} & \cdots & \frac{\partial\alpha_{n+1}}{\partial x_n} & \frac{\partial\alpha_{n+1}}{\partial y} \end{vmatrix}, \tag{A.13}$$

$\{\mathrm{j}_1,\ldots,\mathrm{j}_{n+1}\}$ – orthonormal basis in R^{n+1}.

Consider A_{n+1} as a vector function of y:

$$A_{n+1} = G(y) = \sum_{j=1}^{n+1} \mathfrak{j}_j \times g_j(y) = (g_1(y), \ldots, g_{n+1}(y)). \tag{A.14}$$

Taking into account (A.11) one can see, that $g_j(y)$, $j = 1 \ldots n+1$, are polinomials of y of degree at most $n-1$.

We argue, that $G(y) = 0$ for $y = t$ and $y = x_i$, $i = 3, \ldots, n$.

From (A.11) one can see, that for $k = 1, \ldots, n+1$

$$\alpha_k = (-1)^{k-1}(t^2(\sigma_{k-2} + y\sigma_{k-3}) + 2t(\sigma_{k-1} + y\sigma_{k-2}) + \sigma_k + y\sigma_{k-1}),$$

where σ_i were defined earlier for $i = 1, \ldots, n-2$, $\sigma_0 = 1$, $\sigma_m = 0$ for $m < 0$ and $m > n-2$.

Then

$$\frac{\partial \alpha_k}{\partial y} = (-1)^{k-1}(t^2\sigma_{k-3} + 2t\sigma_{k-2} + \sigma_{k-1}),$$

$$\frac{\partial \alpha_k}{\partial t} = (-1)^{k-1}2(t(\sigma_{k-2} + y\sigma_{k-3}) + \sigma_{k-1} + y\sigma_{k-2}),$$

$$\frac{\partial \alpha_k}{\partial x_r} = (-1)^{k-1}\left(t^2\left(\frac{\partial \sigma_{k-2}}{\partial x_r} + y\frac{\partial \sigma_{k-3}}{\partial x_r}\right) + 2t\left(\frac{\partial \sigma_{k-1}}{\partial x_r} + y\frac{\partial \sigma_{k-2}}{\partial x_r}\right) + \frac{\partial \sigma_k}{\partial x_r} + y\frac{\partial \sigma_{k-1}}{\partial x_r}\right),$$

where $r = n-3, \ldots, n$.

For $\frac{\partial \sigma_i}{\partial x_r}$ holds equality

$$\frac{\partial \sigma_i}{\partial x_r} = \sigma_{i-1}^{(r)},$$

where $\sigma_j^{(r)}$– elementary symmetric polinomial of degree j of $n-3$ variables $x_3, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n$.

Then

$$\frac{\partial \alpha_k}{\partial x_r} = (-1)^{k-1}\left(t^2(\sigma_{k-3}^{(r)} + y\sigma_{k-4}^{(r)}) + 2t(\sigma_{k-2}^{(r)} + y\sigma_{k-3}^{(r)}) + \sigma_{k-1}^{(r)} + y\sigma_{k-2}^{(r)}\right). \tag{A.15}$$

For σ_i and $\sigma_i^{(r)}$ holds equality

$$\sigma_i = \sigma_i^{(r)} + x_r\sigma_{i-1}^{(r)}. \tag{A.16}$$

Then on the basis of (A.15) and (A.16) one can see, that $\forall k = 1, \ldots, n+1$ holds equality

$$\left.\frac{\partial \alpha_k}{\partial x_r}\right|_{y=x_r} = (-1)^{k-1}(t^2\sigma_{k-3} + 2t\sigma_{k-2} + \sigma_{k-1}) = \frac{\partial \alpha_k}{\partial y}.$$

It means, that for $y = x_r$ column r is equal to column $n+1$ in determinant A_{n+1} (A.13), $r = 3, \ldots, n$.

Moreover,

$$\left.\frac{\partial \alpha_k}{\partial t}\right|_{y=t} = (-1)^{k-1}(t^2\sigma_{k-3} + 2t\sigma_{k-2} + \sigma_{k-1}) = \frac{\partial \alpha_k}{\partial y}.$$

It means, that for $y = t$ column 2 is equal to column $n+1$ in determinant A_{n+1} (A.13).

Thus,

$$G(y) = K\ (t - y) \prod_{3\leqslant i\leqslant n} (x_i - y), \tag{A.17}$$

where K – vector in R^{n+1}.

Then

$$A_{n+1}|_{y=0} = G(0) = K t x_3 \cdots x_n, \tag{A.18}$$

where $A_{n+1}|_{y=0}$ – determinant A_{n+1} in (A.13) with $y = 0$.

Determinant $A_{n+1}|_{y=0}$ can be expanded along the last row, taking into account, that only the first and the last elements in that row will remain non-zero. Then

$$A_{n+1}|_{y=0} = \mathrm{j}_{n+1} B_n + A_n \left.\frac{\partial \alpha_{n+1}}{\partial y}\right|_{y=0}$$

and

$$\| A_{n+1}|_{y=0}\| = \sqrt{(B_n)^2 + \|A_n\|^2 (t^2 x_3 \cdots x_n)^2}, \tag{A.19}$$

where $\|A_n\|$ was defined in (A.10), B_n – cofactor of j_{n+1} in expansion of $A_{n+1}|_{y=0}$.

Taking into account (A.11),

$$B_n \simeq$$

$$\begin{vmatrix} 2 & 1 & \cdots & 1 & 1 \\ 2t+2\sigma_1 & 2t+\sigma_1^{(3)} & \cdots & 2t+\sigma_1^{(n)} & 2t+\sigma_1 \\ 2t\sigma_1+2\sigma_2 & t^2+2t\sigma_1^{(3)}+\sigma_2^{(3)} & \cdots & t^2+2t\sigma_1^{(n)}+\sigma_2^{(n)} & t^2+2t\sigma_1+\sigma_2 \\ 2t\sigma_2+2\sigma_3 & t^2\sigma_1^{(3)}+2t\sigma_2^{(3)}+\sigma_3^{(3)} & \cdots & t^2\sigma_1^{(n)}+2t\sigma_2^{(n)}+\sigma_3^{(n)} & t^2\sigma_1+2t\sigma_2+\sigma_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2t\sigma_{n-4}+2\sigma_{n-3} & t^2\sigma_{n-5}^{(3)}+2t\sigma_{n-4}^{(3)}+\sigma_{n-3}^{(3)} & \cdots & t^2\sigma_{n-5}^{(n)}+2t\sigma_{n-4}^{(n)}+\sigma_{n-3}^{(n)} & t^2\sigma_{n-5}+2t\sigma_{n-4}+\sigma_{n-3} \\ 2t\sigma_{n-3}+2\sigma_{n-2} & t^2\sigma_{n-4}^{(3)}+2t\sigma_{n-3}^{(3)} & \cdots & t^2\sigma_{n-4}^{(n)}+2t\sigma_{n-3}^{(n)} & t^2\sigma_{n-4}+2t\sigma_{n-3}+\sigma_{n-2} \\ 2t\sigma_{n-2} & t^2\sigma_{n-3}^{(3)} & \cdots & t^2\sigma_{n-3}^{(n)} & t^2\sigma_{n-3}+2t\sigma_{n-2} \end{vmatrix}. \tag{A.20}$$

Lemma 1.

$$B_n \simeq 2t x_3 \cdots x_n \left(\prod_{3\leqslant i\leqslant n} |t - x_i|\right)\left(\prod_{3\leqslant i<j\leqslant n} |x_i - x_j|\right). \tag{A.21}$$

The proof of Lemma 1 is presented in the end of the proof of the proposition.

Now using (A.19), (A.10) and (A.21) in Lemma 1, one can derive

$$\| A_{n+1}|_{y=0}\| = 2|t x_3 \cdots x_n| \left(\prod_{3\leqslant i\leqslant n} |t - x_i|\right)\left(\prod_{3\leqslant i<j\leqslant n} |x_i - x_j|\right)\sqrt{1 + t^2(1 + t^2 + \ldots + t^{2(n-1)})}$$

and on the basis of (A.18) we have

$$\|K\| = \frac{\| A_{n+1}|_{y=0}\|}{|t x_3 \cdots x_n|}. \tag{A.22}$$

Bringing back the notation $x_{n+1} = y$, on the basis of (A.17) and (A.22) one can obtain

$$\|G(x_{n+1})\| = 2\left(\prod_{3\leqslant i\leqslant n+1} |t - x_i|\right)\left(\prod_{3\leqslant i<j\leqslant n+1} |x_i - x_j|\right)\sqrt{1 + t^2 + \ldots + t^{2n}}. \tag{A.23}$$

And on the basis of (A.12), (A.14) and (A.23) one can obtain required induction step

$$S(Surf_{n+1}) \tag{A.24}$$

$$= 2 \underset{\substack{t\in[-1,1] \\ -1\leqslant x_3\leqslant\ldots\leqslant x_n\leqslant x_{n+1}\leqslant 1}}{\int\cdots\int} \sqrt{1+t^2+\ldots+t^{2n}}\left(\prod_{3\leqslant i\leqslant n+1}|t-x_i|\right)\left(\prod_{3\leqslant i<j\leqslant n+1}|x_i-x_j|\right) dt\, dx_3 \ldots dx_n\, dx_{n+1}.$$

A.2.1. Proof of Lemma 1

With standard computation methods one can check, that lemma is true for $n = 3, 4$.

$$B_3 \simeq \begin{vmatrix} 2 & 1 & 1 \\ 2t + 2x_3 & 2t & 2t + x_3 \\ 2tx_3 & t^2 & t^2 + 2tx_3 \end{vmatrix} \simeq 2tx_3(t - x_3),$$

$$B_4 \simeq \begin{vmatrix} 2 & 1 & 1 & 1 \\ 2t + 2(x_3 + x_4) & 2t + x_4 & 2t + x_3 & 2t + x_3 + x_4 \\ 2t(x_3 + x_4) + 2x_3x_4 & t^2 + 2tx_4 & t^2 + 2tx_3 & t^2 + 2t(x_3 + x_4) + x_3x_4 \\ 2tx_3x_4 & t^2x_4 & t^2x_3 & t^2(x_3 + x_4) + 2tx_3x_4 \end{vmatrix}$$
$$\simeq 2tx_3x_4(t - x_3)(t - x_4)(x_3 - x_4).$$

Now we will prove lemma for arbitrary n.

Meanwhile, one can find compact form of the idea of the proof in "II. Solution by R.J. Walker" [7], where authors prove similar statement.

First step.

On the basis of (A.16) make substitutions in (A.20) in rows $2, \ldots, n$ in columns $2, \ldots, n-1$. Obtain

$$B_n \simeq 2*$$
$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ t + \sigma_1 & 2t + \sigma_1 - x_3 & \cdots & 2t + \sigma_1 - x_n & 2t + \sigma_1 \\ t\sigma_1 + \sigma_2 & t^2 + 2t(\sigma_1 - x_3) + \sigma_2 - x_3\sigma_1^{(3)} & \cdots & t^2 + 2t(\sigma_1 - x_n) + \sigma_2 - x_n\sigma_1^{(n)} & t^2 + 2t\sigma_1 + \sigma_2 \\ t\sigma_2 + \sigma_3 & t^2(\sigma_1 - x_3) + 2t(\sigma_2 - x_3\sigma_1^{(3)}) + \sigma_3 - x_3\sigma_2^{(3)} & \cdots & t^2(\sigma_1 - x_n) + 2t(\sigma_2 - x_n\sigma_1^{(n)}) + \sigma_3 - x_n\sigma_2^{(n)} & t^2\sigma_1 + 2t\sigma_2 + \sigma_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t\sigma_{n-3} + \sigma_{n-2} & t^2(\sigma_{n-4} - x_3\sigma_{n-5}^{(3)}) + 2t(\sigma_{n-3} - x_3\sigma_{n-4}^{(3)}) & \cdots & t^2(\sigma_{n-4} - x_n\sigma_{n-5}^{(n)}) + 2t(\sigma_{n-3} - x_n\sigma_{n-4}^{(n)}) & t^2\sigma_{n-4} + 2t\sigma_{n-3} + \sigma_{n-2} \\ t\sigma_{n-2} & t^2(\sigma_{n-3} - x_3\sigma_{n-4}^{(3)}) & \cdots & t^2(\sigma_{n-3} - x_n\sigma_{n-4}^{(n)}) & t^2\sigma_{n-3} + 2t\sigma_{n-2} \end{vmatrix}.$$

From rows $j = 2, \ldots, n-1$ subtract the first row, multiplied by the last element of jth row $b_{j,n}$. From row n subtract the first row, multiplied by $t^2\sigma_{n-3}$. Multiply rows $j = 2, \ldots, n$ by -1.

Obtain

$$B_n \simeq 2*
\begin{vmatrix}
1 & 1 & \cdots & 1 & 1 \\
t & x_3 & \cdots & x_n & 0 \\
t^2+t\sigma_1 & 2tx_3+x_3\sigma_1^{(3)} & \cdots & 2tx_n+x_n\sigma_1^{(n)} & 0 \\
t^2\sigma_1+t\sigma_2 & t^2x_3+2tx_3\sigma_1^{(3)}+x_3\sigma_2^{(3)} & \cdots & t^2x_n+2tx_n\sigma_1^{(n)}+x_n\sigma_2^{(n)} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
t^2\sigma_{n-5}+t\sigma_{n-4} & t^2x_3\sigma_{n-6}^{(3)}+2tx_3\sigma_{n-5}^{(3)}+x_3\sigma_{n-4}^{(3)} & \cdots & t^2x_n\sigma_{n-6}^{(n)}+2tx_n\sigma_{n-5}^{(n)}+x_n\sigma_{n-4}^{(n)} & 0 \\
t^2\sigma_{n-4}+t\sigma_{n-3} & t^2x_3\sigma_{n-5}^{(3)}+2tx_3\sigma_{n-4}^{(3)}+\sigma_{n-2} & \cdots & t^2x_n\sigma_{n-5}^{(n)}+2tx_n\sigma_{n-4}^{(n)}+\sigma_{n-2} & 0 \\
t^2\sigma_{n-3}-t\sigma_{n-2} & t^2x_3\sigma_{n-4}^{(3)} & \cdots & t^2x_n\sigma_{n-4}^{(n)} & -2t\sigma_{n-2}
\end{vmatrix}.$$

In row $n-1$ in columns $j=2,\ldots,n-1$ apply identity $\sigma_{n-2}=x_{j+1}\sigma_{n-3}^{(j+1)}$. Obtain

$$B_n \simeq 2*
\begin{vmatrix}
1 & 1 & \cdots & 1 & 1 \\
t & x_3 & \cdots & x_n & 0 \\
t^2+t\sigma_1 & 2tx_3+x_3\sigma_1^{(3)} & \cdots & 2tx_n+x_n\sigma_1^{(n)} & 0 \\
t^2\sigma_1+t\sigma_2 & t^2x_3+2tx_3\sigma_1^{(3)}+x_3\sigma_2^{(3)} & \cdots & t^2x_n+2tx_n\sigma_1^{(n)}+x_n\sigma_2^{(n)} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
t^2\sigma_{n-5}+t\sigma_{n-4} & t^2x_3\sigma_{n-6}^{(3)}+2tx_3\sigma_{n-5}^{(3)}+x_3\sigma_{n-4}^{(3)} & \cdots & t^2x_n\sigma_{n-6}^{(n)}+2tx_n\sigma_{n-5}^{(n)}+x_n\sigma_{n-4}^{(n)} & 0 \\
t^2\sigma_{n-4}+t\sigma_{n-3} & t^2x_3\sigma_{n-5}^{(3)}+2tx_3\sigma_{n-4}^{(3)}+x_3\sigma_{n-3}^{(3)} & \cdots & t^2x_n\sigma_{n-5}^{(n)}+2tx_n\sigma_{n-4}^{(n)}+x_n\sigma_{n-3}^{(n)} & 0 \\
t^2\sigma_{n-3}-t\sigma_{n-2} & t^2x_3\sigma_{n-4}^{(3)} & \cdots & t^2x_n\sigma_{n-4}^{(n)} & -2t\sigma_{n-2}
\end{vmatrix}.$$

Take factor t out of the first column, then take factors x_{j+1} out of columns $j=2,\ldots,n-1$ respectively. Obtain

$$B_n \simeq 2tx_3\cdots x_n*
\begin{vmatrix}
t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 1 \\
1 & 1 & \cdots & 1 & 0 \\
t+\sigma_1 & 2t+\sigma_1^{(3)} & \cdots & 2t+\sigma_1^{(n)} & 0 \\
t\sigma_1+\sigma_2 & t^2+2t\sigma_1^{(3)}+\sigma_2^{(3)} & \cdots & t^2+2t\sigma_1^{(n)}+\sigma_2^{(n)} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
t\sigma_{n-5}+\sigma_{n-4} & t^2\sigma_{n-6}^{(3)}+2t\sigma_{n-5}^{(3)}+\sigma_{n-4}^{(3)} & \cdots & t^2\sigma_{n-6}^{(n)}+2t\sigma_{n-5}^{(n)}+\sigma_{n-4}^{(n)} & 0 \\
t\sigma_{n-4}+\sigma_{n-3} & t^2\sigma_{n-5}^{(3)}+2t\sigma_{n-4}^{(3)}+\sigma_{n-3}^{(3)} & \cdots & t^2\sigma_{n-5}^{(n)}+2t\sigma_{n-4}^{(n)}+\sigma_{n-3}^{(n)} & 0 \\
t\sigma_{n-3}-\sigma_{n-2} & t^2\sigma_{n-4}^{(3)} & \cdots & t^2\sigma_{n-4}^{(n)} & (-1)^1 2t\sigma_{n-2}
\end{vmatrix}.$$

Second step.

On the basis of (A.16) make substitutions in rows $3, \ldots, n$ and columns $2, \ldots, n-1$. Obtain

$$B_n \simeq 2tx_3 \cdots x_n *$$
$$\begin{vmatrix} t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 1 \\ 1 & 1 & \cdots & 1 & 0 \\ t+\sigma_1 & 2t+\sigma_1-x_3 & \cdots & 2t+\sigma_1-x_n & 0 \\ t\sigma_1+\sigma_2 & t^2+2t(\sigma_1-x_3)+\sigma_2-x_3\sigma_1^{(3)} & \cdots & t^2+2t(\sigma_1-x_n)+\sigma_2-x_n\sigma_1^{(n)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t\sigma_{n-5}+\sigma_{n-4} & t^2(\sigma_{n-6}-x_3\sigma_{n-7}^{(3)})+2t(\sigma_{n-5}-x_3\sigma_{n-6}^{(3)})+\sigma_{n-4}-x_3\sigma_{n-5}^{(3)} & \cdots & t^2(\sigma_{n-6}-x_n\sigma_{n-7}^{(n)})+2t(\sigma_{n-5}-x_n\sigma_{n-6}^{(n)})+\sigma_{n-4}-x_n\sigma_{n-5}^{(n)} & 0 \\ t\sigma_{n-4}+\sigma_{n-3} & t^2(\sigma_{n-5}-x_3\sigma_{n-6}^{(3)})+2t(\sigma_{n-4}-x_3\sigma_{n-5}^{(3)})+\sigma_{n-3}-x_3\sigma_{n-4}^{(3)} & \cdots & t^2(\sigma_{n-5}-x_n\sigma_{n-6}^{(n)})+2t(\sigma_{n-4}-x_n\sigma_{n-5}^{(n)})+\sigma_{n-3}-x_n\sigma_{n-4}^{(n)} & 0 \\ t\sigma_{n-3}-\sigma_{n-2} & t^2(\sigma_{n-4}-x_3\sigma_{n-5}^{(3)}) & \cdots & t^2(\sigma_{n-4}-x_n\sigma_{n-5}^{(n)}) & -2t\sigma_{n-2} \end{vmatrix}.$$

From row 3 subtract row 2, multiplied by $2t + \sigma_1$. From row 4 subtract row 2, multiplied by $t^2 + 2t\sigma_1 + \sigma_2$. From rows $j = 5, \ldots, n-1$ subtract row 2, multiplied by $t^2\sigma_{j-4} + 2t\sigma_{j-3} + \sigma_{j-2}$. From row n subtract row 2, multiplied by $t^2\sigma_{n-4} - \sigma_{n-2}$. Multiply rows $j = 3, \ldots, n$ by -1. Obtain

$$B_n \simeq 2tx_3 \cdots x_n *$$
$$\begin{vmatrix} t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 1 \\ 1 & 1 & \cdots & 1 & 0 \\ t & x_3 & \cdots & x_n & 0 \\ t^2+t\sigma_1 & 2tx_3+x_3\sigma_1^{(3)} & \cdots & 2tx_n+x_n\sigma_1^{(n)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^2\sigma_{n-6}+t\sigma_{n-5} & t^2x_3\sigma_{n-7}^{(3)}+2tx_3\sigma_{n-6}^{(3)}+x_3\sigma_{n-5}^{(3)} & \cdots & t^2x_n\sigma_{n-7}^{(n)}+2tx_n\sigma_{n-6}^{(n)}+x_n\sigma_{n-5}^{(n)} & 0 \\ t^2\sigma_{n-5}+t\sigma_{n-4} & t^2x_3\sigma_{n-6}^{(3)}+2tx_3\sigma_{n-5}^{(3)}+x_3\sigma_{n-4}^{(3)} & \cdots & t^2x_n\sigma_{n-6}^{(n)}+2tx_n\sigma_{n-5}^{(n)}+x_n\sigma_{n-4}^{(n)} & 0 \\ t^2\sigma_{n-4}-t\sigma_{n-3} & t^2x_3\sigma_{n-5}^{(3)}-\sigma_{n-2} & \cdots & t^2x_n\sigma_{n-5}^{(n)}-\sigma_{n-2} & 2t\sigma_{n-2} \end{vmatrix}.$$

In row n in columns $j = 2, \ldots, n-1$ apply identity $\sigma_{n-2} = x_{j+1}\sigma_{n-3}^{(j+1)}$. Obtain

$$B_n \simeq 2tx_3 \cdots x_n *$$
$$\begin{vmatrix} t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 1 \\ 1 & 1 & \cdots & 1 & 0 \\ t & x_3 & \cdots & x_n & 0 \\ t^2+t\sigma_1 & 2tx_3+x_3\sigma_1^{(3)} & \cdots & 2tx_n+x_n\sigma_1^{(n)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^2\sigma_{n-6}+t\sigma_{n-5} & t^2x_3\sigma_{n-7}^{(3)}+2tx_3\sigma_{n-6}^{(3)}+x_3\sigma_{n-5}^{(3)} & \cdots & t^2x_n\sigma_{n-7}^{(n)}+2tx_n\sigma_{n-6}^{(n)}+x_n\sigma_{n-5}^{(n)} & 0 \\ t^2\sigma_{n-5}+t\sigma_{n-4} & t^2x_3\sigma_{n-6}^{(3)}+2tx_3\sigma_{n-5}^{(3)}+x_3\sigma_{n-4}^{(3)} & \cdots & t^2x_n\sigma_{n-6}^{(n)}+2tx_n\sigma_{n-5}^{(n)}+x_n\sigma_{n-4}^{(n)} & 0 \\ t^2\sigma_{n-4}-t\sigma_{n-3} & t^2x_3\sigma_{n-5}^{(3)}-x_3\sigma_{n-3}^{(3)} & \cdots & t^2x_n\sigma_{n-5}^{(n)}-x_n\sigma_{n-3}^{(n)} & 2t\sigma_{n-2} \end{vmatrix}.$$

Take factor t out of the first column, then take factors x_{j+1} out of columns $j = 2, \ldots, n-1$ respectively. Obtain

$$B_n \simeq 2t^2 x_3^2 \cdots x_n^2 * \begin{vmatrix} t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 1 \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ t+\sigma_1 & 2t+\sigma_1^{(3)} & \cdots & 2t+\sigma_1^{(n)} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t\sigma_{n-6}+\sigma_{n-5} & t^2\sigma_{n-7}^{(3)}+2t\sigma_{n-6}^{(3)}+\sigma_{n-5}^{(3)} & \cdots & t^2\sigma_{n-7}^{(n)}+2t\sigma_{n-6}^{(n)}+\sigma_{n-5}^{(n)} & 0 \\ t\sigma_{n-5}+\sigma_{n-4} & t^2\sigma_{n-6}^{(3)}+2t\sigma_{n-5}^{(3)}+\sigma_{n-4}^{(3)} & \cdots & t^2\sigma_{n-6}^{(n)}+2t\sigma_{n-5}^{(n)}+\sigma_{n-4}^{(n)} & 0 \\ t\sigma_{n-4}-\sigma_{n-3} & t^2\sigma_{n-5}^{(3)}-\sigma_{n-3}^{(3)} & \cdots & t^2\sigma_{n-5}^{(n)}-\sigma_{n-3}^{(n)} & (-1)^2 2t\sigma_{n-2} \end{vmatrix}.$$

Third step.

On the basis of (A.16) make substitutions in rows $4, \ldots, n$ and columns $2, \ldots, n-1$. Obtain

$$B_n \simeq 2t^2 x_3^2 \cdots x_n^2 * \begin{vmatrix} t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 1 \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ t+\sigma_1 & 2t+\sigma_1-x_3 & \cdots & 2t+\sigma_1-x_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t\sigma_{n-6}+\sigma_{n-5} & t^2(\sigma_{n-7}-x_3\sigma_{n-8}^{(3)})+2t(\sigma_{n-6}-x_3\sigma_{n-7}^{(3)})+\sigma_{n-5}-x_3\sigma_{n-6}^{(3)} & \cdots & t^2(\sigma_{n-7}-x_n\sigma_{n-8}^{(n)})+2t(\sigma_{n-6}-x_n\sigma_{n-7}^{(n)})+\sigma_{n-5}-x_n\sigma_{n-6}^{(n)} & 0 \\ t\sigma_{n-5}+\sigma_{n-4} & t^2(\sigma_{n-6}-x_3\sigma_{n-7}^{(3)})+2t(\sigma_{n-5}-x_3\sigma_{n-6}^{(3)})+\sigma_{n-4}-x_3\sigma_{n-5}^{(3)} & \cdots & t^2(\sigma_{n-6}-x_n\sigma_{n-7}^{(n)})+2t(\sigma_{n-5}-x_n\sigma_{n-6}^{(n)})+\sigma_{n-4}-x_n\sigma_{n-5}^{(n)} & 0 \\ t\sigma_{n-4}-\sigma_{n-3} & t^2(\sigma_{n-5}-x_3\sigma_{n-6}^{(3)})-(\sigma_{n-3}-x_3\sigma_{n-4}^{(3)}) & \cdots & t^2(\sigma_{n-5}-x_n\sigma_{n-6}^{(n)})-(\sigma_{n-3}-x_n\sigma_{n-4}^{(n)}) & 2t\sigma_{n-2} \end{vmatrix}.$$

From row 4 subtract row 3, multiplied by $2t + \sigma_1$. From row 5 subtract row 3, multiplied by $t^2 + 2t\sigma_1 + \sigma_2$. From rows $j = 6, \ldots, n-1$ subtract row 3, multiplied by $t^2\sigma_{j-5} + 2t\sigma_{j-4} + \sigma_{j-3}$. From row n subtract row 3, multiplied by $t^2\sigma_{n-5} - \sigma_{n-3}$. Multiply rows $j = 4, \ldots, n$ by -1. Obtain

$$B_n \simeq 2t^2 x_3^2 \cdots x_n^2 * \begin{vmatrix} t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 1 \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ t & x_3 & \cdots & x_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^2\sigma_{n-7}+t\sigma_{n-6} & t^2x_3\sigma_{n-8}^{(3)}+2tx_3\sigma_{n-7}^{(3)}+x_3\sigma_{n-6}^{(3)} & \cdots & t^2x_n\sigma_{n-8}^{(n)}+2tx_n\sigma_{n-7}^{(n)}+x_n\sigma_{n-6}^{(n)} & 0 \\ t^2\sigma_{n-6}+t\sigma_{n-5} & t^2x_3\sigma_{n-7}^{(3)}+2tx_3\sigma_{n-6}^{(3)}+x_3\sigma_{n-5}^{(3)} & \cdots & t^2x_n\sigma_{n-7}^{(n)}+2tx_n\sigma_{n-6}^{(n)}+x_n\sigma_{n-5}^{(n)} & 0 \\ t^2\sigma_{n-5}-t\sigma_{n-4} & t^2x_3\sigma_{n-6}^{(3)}-x_3\sigma_{n-4}^{(3)} & \cdots & t^2x_n\sigma_{n-6}^{(n)}-x_n\sigma_{n-4}^{(n)} & -2t\sigma_{n-2} \end{vmatrix}.$$

Take factor t out of the first column, then take factors x_{j+1} out of columns $j = 2, \ldots, n-1$ respectively. Obtain

$$B_n \simeq 2t^3 x_3^3 \cdots x_n^3 *$$
$$\begin{vmatrix} t^{-3} & x_3^{-3} & \cdots & x_n^{-3} & 1 \\ t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 0 \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t\sigma_{n-7} + \sigma_{n-6} & t^2\sigma_{n-8}^{(3)} + 2t\sigma_{n-7}^{(3)} + \sigma_{n-6}^{(3)} & \cdots & t^2\sigma_{n-8}^{(n)} + 2t\sigma_{n-7}^{(n)} + \sigma_{n-6}^{(n)} & 0 \\ t\sigma_{n-6} + \sigma_{n-5} & t^2\sigma_{n-7}^{(3)} + 2t\sigma_{n-6}^{(3)} + \sigma_{n-5}^{(3)} & \cdots & t^2\sigma_{n-7}^{(n)} + 2t\sigma_{n-6}^{(n)} + \sigma_{n-5}^{(n)} & 0 \\ t\sigma_{n-5} - \sigma_{n-4} & t^2\sigma_{n-6}^{(3)} - \sigma_{n-4}^{(3)} & \cdots & t^2\sigma_{n-6}^{(n)} - \sigma_{n-4}^{(n)} & (-1)^3 2t\sigma_{n-2} \end{vmatrix}.$$

The third step is finished.

Now perform steps $4, \ldots, n-5$ in the same way, as step 3.

After finishing step $n-5$ obtain

$$B_n \simeq 2t^{n-5} x_3^{n-5} \cdots x_n^{n-5} *$$
$$\begin{vmatrix} t^{-n+5} & x_3^{-n+5} & \cdots & x_n^{-n+5} & 1 \\ t^{-n+6} & x_3^{-n+6} & \cdots & x_n^{-n+6} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ t + \sigma_1 & 2t + \sigma_1^{(3)} & \cdots & 2t + \sigma_1^{(n)} & 0 \\ t\sigma_1 + \sigma_2 & t^2 + 2t\sigma_1^{(3)} + \sigma_2^{(3)} & \cdots & t^2 + 2t\sigma_1^{(n)} + \sigma_2^{(n)} & 0 \\ t\sigma_2 + \sigma_3 & t^2\sigma_1^{(3)} + 2t\sigma_2^{(3)} + \sigma_3^{(3)} & \cdots & t^2\sigma_1^{(n)} + 2t\sigma_2^{(n)} + \sigma_3^{(n)} & 0 \\ t\sigma_3 - \sigma_4 & t^2\sigma_2^{(3)} - \sigma_4^{(3)} & \cdots & t^2\sigma_2^{(n)} - \sigma_4^{(n)} & (-1)^{n-5} 2t\sigma_{n-2} \end{vmatrix}.$$

$(n-4)$th step.

On the basis of (A.16) make substitutions in rows $(n-3), \ldots, n$ in columns $2, \ldots, n-1$. Obtain

$$B_n \simeq 2t^{n-5} x_3^{n-5} \cdots x_n^{n-5} *$$
$$\begin{vmatrix} t^{-n+5} & x_3^{-n+5} & \cdots & x_n^{-n+5} & 1 \\ t^{-n+6} & x_3^{-n+6} & \cdots & x_n^{-n+6} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ t + \sigma_1 & 2t + \sigma_1 - x_3 & \cdots & 2t + \sigma_1 - x_n & 0 \\ t\sigma_1 + \sigma_2 & t^2 + 2t(\sigma_1 - x_3) + \sigma_2 - x_3\sigma_1^{(3)} & \cdots & t^2 + 2t(\sigma_1 - x_n) + \sigma_2 - x_n\sigma_1^{(n)} & 0 \\ t\sigma_2 + \sigma_3 & t^2(\sigma_1 - x_3) + 2t(\sigma_2 - x_3\sigma_1^{(3)}) + \sigma_3 - x_3\sigma_2^3 & \cdots & t^2(\sigma_1 - x_n) + 2t(\sigma_2 - x_n\sigma_1^{(n)}) + \sigma_3 - x_n\sigma_2^{(n)} & 0 \\ t\sigma_3 - \sigma_4 & t^2(\sigma_2 - x_3\sigma_1^{(3)}) - (\sigma_4 - x_3\sigma_3^{(3)}) & \cdots & t^2(\sigma_2 - x_n\sigma_1^{(n)}) - (\sigma_4 - x_n\sigma_3^{(n)}) & (-1)^{n-5} 2t\sigma_{n-2} \end{vmatrix}.$$

From row $n-3$ subtract row $n-4$, multiplied by $2t+\sigma_1$. From row $n-2$ subtract row $n-4$, multiplied by $t^2+2t\sigma_1+\sigma_2$. From row $n-1$ subtract row $n-4$, multiplied by $t^2\sigma_1+2t\sigma_2+\sigma_3$. From row n subtract row $n-4$, multiplied by $t^2\sigma_2-\sigma_4$. Multiply rows $j=n-3,\ldots,n$ by -1. Obtain

$$B_n \simeq 2t^{n-5}x_3^{n-5}\cdots x_n^{n-5} *$$
$$\begin{vmatrix}
t^{-n+5} & x_3^{-n+5} & \cdots & x_n^{-n+5} & 1 \\
t^{-n+6} & x_3^{-n+6} & \cdots & x_n^{-n+6} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\
1 & 1 & \cdots & 1 & 0 \\
t & x_3 & \cdots & x_n & 0 \\
t^2+t\sigma_1 & 2tx_3+x_3\sigma_1^{(3)} & \cdots & 2tx_n+x_n\sigma_1^{(n)} & 0 \\
t^2\sigma_1+t\sigma_2 & t^2x_3+2tx_3\sigma_1^{(3)}+x_3\sigma_2^3 & \cdots & t^2x_n+2tx_n\sigma_1^{(n)}+x_n\sigma_2^{(n)} & 0 \\
t^2\sigma_2-t\sigma_3 & t^2x_3\sigma_1^{(3)}-x_3\sigma_3^{(3)} & \cdots & t^2x_n\sigma_1^{(n)}-x_n\sigma_3^{(n)} & (-1)^{n-4}2t\sigma_{n-2}
\end{vmatrix}.$$

Take factor t out of the first column, then take factors x_{j+1} out of columns $j=2,\ldots,n-1$ respectively. Obtain

$$B_n \simeq 2t^{n-4}x_3^{n-4}\cdots x_n^{n-4} *$$
$$\begin{vmatrix}
t^{-n+4} & x_3^{-n+4} & \cdots & x_n^{-n+4} & 1 \\
t^{-n+5} & x_3^{-n+5} & \cdots & x_n^{-n+5} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 0 \\
t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\
1 & 1 & \cdots & 1 & 0 \\
t+\sigma_1 & 2t+\sigma_1^{(3)} & \cdots & 2t+\sigma_1^{(n)} & 0 \\
t\sigma_1+\sigma_2 & t^2+2t\sigma_1^{(3)}+\sigma_2^{(3)} & \cdots & t^2+2t\sigma_1^{(n)}+\sigma_2^{(n)} & 0 \\
t\sigma_2-\sigma_3 & t^2\sigma_1^{(3)}-\sigma_3^{(3)} & \cdots & t^2\sigma_1^{(n)}-\sigma_3^{(n)} & (-1)^{n-4}2t\sigma_{n-2}
\end{vmatrix}.$$

$(n-3)$th step.

On the basis of (A.16) make substitutions in rows $(n-2),\ldots,n$ in columns $2,\ldots,n-1$. Obtain

$$B_n \simeq 2t^{n-4}x_3^{n-4}\cdots x_n^{n-4} *$$
$$\begin{vmatrix}
t^{-n+4} & x_3^{-n+4} & \cdots & x_n^{-n+4} & 1 \\
t^{-n+5} & x_3^{-n+5} & \cdots & x_n^{-n+5} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 0 \\
t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\
1 & 1 & \cdots & 1 & 0 \\
t+\sigma_1 & 2t+\sigma_1-x_3 & \cdots & 2t+\sigma_1-x_n & 0 \\
t\sigma_1+\sigma_2 & t^2+2t(\sigma_1-x_3)+\sigma_2-x_3\sigma_1^{(3)} & \cdots & t^2+2t(\sigma_1-x_n)+\sigma_2-x_n\sigma_1^{(n)} & 0 \\
t\sigma_2-\sigma_3 & t^2(\sigma_1-x_3)-(\sigma_3-x_3\sigma_2^{(3)}) & \cdots & t^2(\sigma_1-x_n)-(\sigma_3-x_n\sigma_2^{(n)}) & (-1)^{n-4}2t\sigma_{n-2}
\end{vmatrix}.$$

From row $n-2$ subtract row $n-3$, multiplied by $2t+\sigma_1$. From row $n-1$ subtract row $n-3$, multiplied by $t^2+2t\sigma_1+\sigma_2$. From row n subtract row $n-3$, multiplied by $t^2\sigma_1-\sigma_3$. Multiply rows $j=n-2,n-1,n$ by -1. Obtain

$$B_n \simeq 2t^{n-4}x_3^{n-4}\cdots x_n^{n-4} *$$
$$\begin{vmatrix} t^{-n+4} & x_3^{-n+4} & \cdots & x_n^{-n+4} & 1 \\ t^{-n+5} & x_3^{-n+5} & \cdots & x_n^{-n+5} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 0 \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ t & x_3 & \cdots & x_n & 0 \\ t^2+t\sigma_1 & 2tx_3+x_3\sigma_1^{(3)} & \cdots & 2tx_n+x_n\sigma_1^{(n)} & 0 \\ t^2\sigma_1-t\sigma_2 & t^2x_3-x_3\sigma_2^{(3)} & \cdots & t^2x_n-x_n\sigma_2^{(n)} & (-1)^{n-3}2t\sigma_{n-2} \end{vmatrix}.$$

Take factor t out of the first column, then take factors x_{j+1} out of columns $j=2,\ldots,n-1$ respectively. Obtain

$$B_n \simeq 2t^{n-3}x_3^{n-3}\cdots x_n^{n-3} *$$
$$\begin{vmatrix} t^{-n+3} & x_3^{-n+3} & \cdots & x_n^{-n+3} & 1 \\ t^{-n+4} & x_3^{-n+4} & \cdots & x_n^{-n+4} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^{-3} & x_3^{-3} & \cdots & x_n^{-3} & 0 \\ t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 0 \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ t+\sigma_1 & 2t+\sigma_1^{(3)} & \cdots & 2t+\sigma_1^{(n)} & 0 \\ t\sigma_1-\sigma_2 & t^2-\sigma_2^{(3)} & \cdots & t^2-\sigma_2^{(n)} & (-1)^{n-3}2t\sigma_{n-2} \end{vmatrix}.$$

$(n-2)$th step.

On the basis of (A.16) make substitutions in rows $n-1,n$ in columns $2,\ldots,n-1$. Obtain

$$B_n \simeq 2t^{n-3}x_3^{n-3}\cdots x_n^{n-3} *$$
$$\begin{vmatrix} t^{-n+3} & x_3^{-n+3} & \cdots & x_n^{-n+3} & 1 \\ t^{-n+4} & x_3^{-n+4} & \cdots & x_n^{-n+4} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^{-3} & x_3^{-3} & \cdots & x_n^{-3} & 0 \\ t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 0 \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ t+\sigma_1 & 2t+\sigma_1-x_3 & \cdots & 2t+\sigma_1-x_n & 0 \\ t\sigma_1-\sigma_2 & t^2-(\sigma_2-x_3\sigma_1^{(3)}) & \cdots & t^2-(\sigma_2-x_n\sigma_1^{(n)}) & (-1)^{n-3}2t\sigma_{n-2} \end{vmatrix}.$$

From row $n-1$ subtract row $n-2$, multiplied by $2t+\sigma_1$. From row n subtract row $n-2$, multiplied by $t^2-\sigma_2$. Multiply rows $j=n-1,n$ by -1. Obtain

$$B_n \simeq 2t^{n-3}x_3^{n-3}\cdots x_n^{n-3} *$$
$$\begin{vmatrix} t^{-n+3} & x_3^{-n+3} & \cdots & x_n^{-n+3} & 1 \\ t^{-n+4} & x_3^{-n+4} & \cdots & x_n^{-n+4} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^{-3} & x_3^{-3} & \cdots & x_n^{-3} & 0 \\ t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 0 \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ t & x_3 & \cdots & x_n & 0 \\ t^2-t\sigma_1 & -x_3\sigma_1^{(3)} & \cdots & -x_n\sigma_1^{(n)} & (-1)^{n-2}2t\sigma_{n-2} \end{vmatrix}.$$

Take factor t out of the first column, then take factors x_{j+1} out of columns $j=2,\ldots,n-1$ respectively. Obtain

$$B_n \simeq 2t^{n-2}x_3^{n-2}\cdots x_n^{n-2} *$$
$$\begin{vmatrix} t^{-n+2} & x_3^{-n+2} & \cdots & x_n^{-n+2} & 1 \\ t^{-n+3} & x_3^{-n+3} & \cdots & x_n^{-n+3} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^{-4} & x_3^{-4} & \cdots & x_n^{-4} & 0 \\ t^{-3} & x_3^{-3} & \cdots & x_n^{-3} & 0 \\ t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 0 \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ t-\sigma_1 & -\sigma_1^{(3)} & \cdots & -\sigma_1^{(n)} & (-1)^{n-2}2t\sigma_{n-2} \end{vmatrix}.$$

$(n-1)$th step.

On the basis of (A.16) make substitutions in the last row in columns $2,\ldots,n-1$. Obtain

$$B_n \simeq 2t^{n-2}x_3^{n-2}\cdots x_n^{n-2} *$$
$$\begin{vmatrix} t^{-n+2} & x_3^{-n+2} & \cdots & x_n^{-n+2} & 1 \\ t^{-n+3} & x_3^{-n+3} & \cdots & x_n^{-n+3} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^{-4} & x_3^{-4} & \cdots & x_n^{-4} & 0 \\ t^{-3} & x_3^{-3} & \cdots & x_n^{-3} & 0 \\ t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 0 \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ t-\sigma_1 & x_3-\sigma_1 & \cdots & x_n-\sigma_1 & (-1)^{n-2}2t\sigma_{n-2} \end{vmatrix}.$$

From row n subtract row $n-1$, multiplied by $-\sigma_1$. Obtain

$$B_n \simeq 2t^{n-2}x_3^{n-2}\cdots x_n^{n-2} *$$
$$\begin{vmatrix} t^{-n+2} & x_3^{-n+2} & \cdots & x_n^{-n+2} & 1 \\ t^{-n+3} & x_3^{-n+3} & \cdots & x_n^{-n+3} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^{-4} & x_3^{-4} & \cdots & x_n^{-4} & 0 \\ t^{-3} & x_3^{-3} & \cdots & x_n^{-3} & 0 \\ t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 0 \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & 0 \\ t & x_3 & \cdots & x_n & (-1)^{n-2}2t\sigma_{n-2} \end{vmatrix}.$$

Take factor t out of the first column, take factors x_{j+1} out of columns $j = 2, \ldots, n-1$ respectively. Obtain

$$B_n \simeq 2t^{n-1}x_3^{n-1}\cdots x_n^{n-1} *$$
$$\begin{vmatrix} t^{-n+1} & x_3^{-n+1} & \cdots & x_n^{-n+1} & 1 \\ t^{-n+2} & x_3^{-n+2} & \cdots & x_n^{-n+2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^{-5} & x_3^{-5} & \cdots & x_n^{-5} & 0 \\ t^{-4} & x_3^{-4} & \cdots & x_n^{-4} & 0 \\ t^{-3} & x_3^{-3} & \cdots & x_n^{-3} & 0 \\ t^{-2} & x_3^{-2} & \cdots & x_n^{-2} & 0 \\ t^{-1} & x_3^{-1} & \cdots & x_n^{-1} & 0 \\ 1 & 1 & \cdots & 1 & (-1)^{n-2}2t\sigma_{n-2} \end{vmatrix}.$$

Now we have finished step $(n-1)$, after which return factors, which were taken out, to the corresponding columns of determinant, and obtain

$$B_n \simeq 2\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ t & x_3 & \cdots & x_n & 0 \\ t^2 & x_3^2 & \cdots & x_n^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} & 0 \\ t^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} & (-1)^{n-2}2t\sigma_{n-2} \end{vmatrix}. \tag{A.25}$$

After expansion of determinant (A.25) along the last column obtain

$$B_n \simeq 2\Big((-1)^{n+1}tx_3\cdots x_nV(t, x_3, \ldots, x_n) + (-1)^{2n}(-1)^{n-2}2tx_3\cdots x_nV(t, x_3, \ldots, x_n)\Big), \tag{A.26}$$

where $V(t, x_3, \ldots, x_n)$ – Vandermonde determinant of variables $t, x_3, \ldots, x_n$.

Taking into account different parities of degrees of -1 in (A.26), obtain

$$B_n \simeq 2tx_3\cdots x_nV(t, x_3, \ldots, x_n).$$

Proofs of Lemma 1 and Proposition 2 are finished.

A.3. PROOF OF PROPOSITION 3

In [2] it is shown, that

$$\log_2 V(D_n) = -\left(\frac{n}{2}+\frac{1}{4}\right)\log_2 n + O(n).$$

Hence,

$$\ln V(D_n) = -\left(\frac{n}{2}+\frac{1}{4}\right)\ln n + O(n).$$

In [3] it is shown, that

$$\ln\left(\frac{V(\mathcal{E}_n)}{V(D_n)}\right) = -\frac{\ln 2}{2}n^2 + \frac{1}{8}\ln n + O(1)$$

and

$$V(\mathcal{E}_n) = \underset{-1\leqslant x_1\leqslant\ldots\leqslant x_n\leqslant 1}{\int\cdots\int}\left(\prod_{1\leqslant i<j\leqslant n}|x_i-x_j|\right)dx_1\ldots dx_n = 2^{\frac{n(n+1)}{2}}\prod_{k=1}^{n}\frac{\{(k-1)!\}^2}{(2k-1)!}. \tag{A.27}$$

Thus,

$$\ln V(\mathcal{E}_n) = -\frac{\ln 2}{2}n^2 - \left(\frac{n}{2}+\frac{1}{8}\right)\ln n + O(n). \tag{A.28}$$

On the basis of (A.5) from the proof of proposition 1 we have

$$S(Pl_n^{(1)}) = S(Pl_n^{(-1)}) = \sqrt{n}\,V(\mathcal{E}_{n-1}).$$

On the basis of (7) in Proposition 2 and (A.27) one can see, that

$$S(Surf_n) = 2\underset{\substack{x_1\in[-1,1]\\ -1\leqslant x_2\leqslant\ldots\leqslant x_{n-1}\leqslant 1}}{\int\cdots\int}\sqrt{1+x_1^2+\ldots+x_1^{2(n-1)}}\left(\prod_{1\leqslant i<j\leqslant n-1}|x_i-x_j|\right)dx_1\ldots dx_{n-1}$$

$$\leqslant 2\sqrt{n}\underset{\substack{x_1\in[-1,1]\\ -1\leqslant x_2\leqslant\ldots\leqslant x_{n-1}\leqslant 1}}{\int\cdots\int}\left(\prod_{1\leqslant i<j\leqslant n-1}|x_i-x_j|\right)dx_1\ldots dx_{n-1} = 2\sqrt{n}(n-1)V(\mathcal{E}_{n-1}).$$

Thus, on the basis of (8) one can obtain

$$S(\partial\mathcal{E}_n) \leqslant 2\sqrt{n}\,nV(\mathcal{E}_{n-1}). \tag{A.29}$$

On the basis of (A.28) and (A.29) one can obtain

$$\ln S(\partial\mathcal{E}_n) \leqslant -\frac{\ln 2}{2}n^2 - \frac{n}{2}\ln n + O(n).$$

For large enough n the value of $(-\frac{\ln 2}{2}n^2 - \frac{n}{2}\ln n)$ dominates over $O(n)$, therefore $\lim\limits_{n\to\infty}\ln S(\partial\mathcal{E}_n) = -\infty$, and therefore $\lim\limits_{n\to\infty} S(\partial\mathcal{E}_n) = 0$.

This finishes the proof of the first part of Proposition 3.

Now we will prove the second part of Proposition 3, that maximum value of $S(\partial\mathcal{E}_n)$ is reached for $n = 3$.

From (A.27) follows, that

$$V(\mathcal{E}_n) = V(\mathcal{E}_{n-1})\, 2^n \frac{\{(n-1)!\}^2}{(2n-1)!}, \qquad n \geqslant 2,$$
$$V(\mathcal{E}_1) = 2.$$

Then, taking into account (A.29), we obtain, that

$$S(\partial\mathcal{E}_n) \leqslant F(n), \qquad n \geqslant 2, \tag{A.30}$$

where

$$F(1) = 2,$$
$$F(n) = F(n-1) \times k(n), \qquad n \geqslant 2,$$
$$k(n) = 2^{n-1} \frac{n\sqrt{n}}{(n-1)\sqrt{n-1}} \frac{\{(n-2)!\}^2}{(2n-3)!}.$$

Now we will show, that $k(n) < 1$ for $n \geqslant 4$.

$$k(n) = 2^{n-1} \frac{n\sqrt{n}}{(n-1)\sqrt{n-1}} \frac{1 \times 2 \cdots (n-2)}{(n-1) \times n \cdots (2n-3)}$$
$$= \frac{2\sqrt{n}}{(n-1)\sqrt{n-1}} \times \frac{2 \times 1}{n-1} \times \frac{2 \times 2}{n+1} \times \frac{2 \times 3}{n+2} \cdots \frac{2 \times (n-2)}{2n-3}. \tag{A.31}$$

In (A.31) there are $(n-1)$ multipliers, and all of them are less than 1 for $n \geqslant n_0 = 4$.

It means, that $k(n) < 1$ and $F(n)$ in (A.30) is a decreasing function for $n \geqslant 4$. Therefore, for $n \geqslant 4$

$$S(\partial\mathcal{E}_n) \leqslant F(n) \leqslant F(4) = 2\sqrt{4} \times 4 \times V(\mathcal{E}_3) \approx 5.689.$$

Taking into account (5), the proof of the second part of Proposition 3 is finished.

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